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The effect of buoyancy forces on the boundary-layer flow over a semi-infinite vertical flat plate in a uniform free stream

By J. H. MERKIN
Department of Mathematics, The University of Manchester

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The boundary-layer flow over a semi-infinite vertical flat plate, heated to a constant temperature in a uniform free stream, is discussed in the two cases when the buoyancy forces aid and oppose the development of the boundary layer. In the former case, two series solutions are obtained, one of which is valid near the leading edge and the other is valid asymptotically. An accurate numerical method is used to describe the flow in the region where the series are not valid. In the latter case, a series, valid near the leading edge is obtained and it is extended by a numerical method to the point where the boundary layer is shown to separate.

1. Introduction

The situation discussed in this paper is that of a uniform free stream U flowing along a semi-infinite vertical flat plate, which is fixed with its leading edge horizontal. The plate is heated to a constant temperature T_1 above the ambient temperature T_0. Heat is supplied by diffusion and convection from the plate, and this heating gives rise to buoyant body forces. There are two cases to consider: one when the plate extends vertically upwards, and the other when it extends vertically downwards. In the first case the buoyancy forces act in the direction of the free stream, and in the second case, they oppose the free stream. In both cases, near the leading edge, there is little chance for heat from the plate to be taken into the fluid, and the boundary layer is formed chiefly by the retardation of the free stream, but the effect of the buoyancy forces increases as the boundary layer develops.

We shall discuss first the case when the free stream and the buoyancy forces are in the same direction. In this case the fluid in the boundary layer is accelerated by the buoyancy forces so that these act like a favourable pressure gradient. We will refer to this case as the favourable case. Far downstream, the layer is governed chiefly by the buoyancy forces.

We solve this problem by first obtaining two series expansions, one which holds near the leading edge, and one which holds far downstream (which, in this case, is upwards). A step-by-step numerical method is used to describe the flow in the region where neither of these series holds.

To obtain the series expansions, a transformation is first applied to the boundary-layer equations. The nature of the transformation is dictated by the fact that near the leading edge the situation is essentially that of buoyancy forces modifying the forced convection solution, while asymptotically it is that of the effect of a free stream on the free convection solution. The problems of the small perturbation of the forced convection solution by free convection effects, and that of the small perturbation of the free convection solution by forced convection effects are discussed by Szewczyk (1964). He obtains series expansions to describe the flow in each case, and gives the first three terms in each series for various values of the Prandtl number. There are several criticisms that I would like to make of Szewczyk's paper. First, his series expansions are purely formal and not related to a specific physical situation, and secondly he expands in terms of parameters, but, in actual fact, they are co-ordinate expansions that he obtains. This leads him to ignore an important point in considering the effect of the free stream on the free convection solution. In the situation under consideration here, this leads to an asymptotic expansion in $\xi^{-\frac{1}{2}}$, where

$$\xi = \frac{g\beta\Delta T}{U^2}x$$

(g is the acceleration due to gravity, β is the coefficient of thermal expansion, and $\Delta T = T_1 - T_0$; x is the co-ordinate that measures distance along the plate). Stewartson (1957, 1964) points out a fundamental difficulty in obtaining asymptotic solutions of the boundary-layer equations. Since the boundary-layer equations are parabolic in nature, the velocity and temperature at a particular station depend on the velocity and temperature distribution upstream of this station. This, as Stewartson points out, leads to an arbitrariness being introduced into the asymptotic expansion at some stage, which is due to neglecting the boundary conditions at the leading edge. Stewartson found that, in order to resolve this and obtain a solution which was exponentially small at infinity, he had to include a logarithmic term in the asymptotic expansion. A similar situation arises here. When we come to solve the differential equations governing the term of $O(\xi^{-1})$, we find that there is a complementary function which satisfies all the required boundary conditions, and so arbitrary multiples of this can be added to any particular integral of the equations which also satisfies the boundary conditions. Because of the existence of this complementary function we find that we have to include a term of $O(\log \xi/\xi)$ in the asymptotic expansion in order to obtain a solution for the term of $O(\xi^{-1})$. Szewczyk makes no mention of this in his paper.

An accurate step-by-step method of the problem is obtained from an adaptation of a method given by Terrill (1960). This starts with velocity and temperature profiles at the leading edge and calculates velocity and temperature profiles downstream to an accuracy of four figures.

When the plate extends vertically downwards the buoyancy forces retard the fluid in the boundary layer, so that these act like an adverse pressure gradient. This case will be referred to as the adverse case.

To solve this problem we obtain a series expansion near the leading edge, which is related to the series expansion near the leading edge in the favourable case. A step-by-step numerical method is used to extend the series solution to the point where the flow is shown to separate. The numerical procedure is the same as that used near the leading edge in the favourable case. The numerical integration in this case was carried out to an accuracy of five figures, and where the flow separates, it suggests that the skin friction and the heat transfer become singular.

The methods given in this paper are general ones, but, for the sake of brevity, only the results for the case when the Prandtl number $\sigma = 1$ will be given.

2. Equations of motion

On the assumption that $U^2/a^2 \ll \Delta T/T_0 \ll 1$

(where a is the velocity of sound in the fluid and $\Delta T = T_1 - T_0$), we can neglect heating due to viscous dissipation and take the fluid as incompressible, so that changes in density are important only in producing buoyancy forces. The kinematic viscosity ν and the thermometric conductivity κ can then be taken as constant (Whitham 1963, p. 127). The boundary-layer equations then become

$$\partial u/\partial x + \partial v/\partial y = 0, \tag{1}$$

$$u\,\partial u/\partial x + v\,\partial u/\partial y = \pm g\beta(T - T_0) + \nu\,\partial^2 u/\partial y^2, \tag{2}$$

$$u\,\partial T/\partial x + v\,\partial T/\partial y = \kappa\,\partial^2 T/\partial y^2. \tag{3}$$

x is measured along the plate, $x = 0$ being the leading edge, and y is measured normally outwards. u and v are the velocities in the x and y directions respectively and T is the temperature of the fluid. In (2) we take the $+$ sign when the plate is vertically upwards (favourable case) and the $-$ sign when the plate is vertically downwards (adverse case). The boundary conditions are

$$u = v = 0, \quad T = T_1 \quad \text{on} \quad y = 0,$$

$$u \to U, \quad T \to T_0 \quad \text{as} \quad y \to \infty$$

and

$$u = U, \quad T = T_0 \quad \text{at} \quad x = 0.$$

3. Solution near the leading edge

Near the leading edge the boundary layer is formed mainly by the retardation of the free-stream U by viscosity. The effect of the buoyancy forces increases as the boundary layer develops from the leading edge. This suggests the following transformation:

$$\psi = (2\nu U x)^{\frac{1}{2}} f(\xi, \eta),$$

$$T - T_0 = \Delta T\,\theta(\xi, \eta),$$

where ψ is the stream function and

$$\eta = y\left(\frac{U}{2\nu x}\right)^{\frac{1}{2}},$$

$$\xi = \frac{g\beta\Delta T}{U^2}\,x$$

and $$\Delta T = T_1 - T_0.$$

The boundary-layer equations (2) and (3) become

$$\frac{\partial^3 f}{\partial \eta^3} + f\frac{\partial^2 f}{\partial \eta^2} + 2\xi\left(\pm\theta + \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta \partial \xi}\right) = 0, \tag{4}$$

$$\frac{1}{\sigma}\frac{\partial^2 \theta}{\partial \eta^2} + f\frac{\partial \theta}{\partial \eta} + 2\xi\left(\frac{\partial f}{\partial \xi}\frac{\partial \theta}{\partial \eta} - \frac{\partial f}{\partial \eta}\frac{\partial \theta}{\partial \xi}\right) = 0, \tag{5}$$

with boundary conditions

$$f = \partial f/\partial \eta = 0, \quad \theta = 1 \quad \text{on} \quad \eta = 0,$$

$$\partial f/\partial \eta \to 1, \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty.$$

$\sigma = \nu/\kappa$ is the Prandtl number.

The method of solution of (4) and (5) is to expand f and θ in series in ξ in the form

$$f = f_0(\eta) \pm \xi f_1(\eta) + \xi^2 f_2(\eta) \pm \ldots, \tag{6}$$

$$\theta = \theta_0(\eta) \pm \xi\theta_1(\eta) + \xi^2\theta_2(\eta) \pm \ldots. \tag{7}$$

On putting these expansions in (4) and (5) and equating powers of ξ, we get essentially the same set of equations given by Szewczyk (1964) (equations (7)–(9) in his paper), except that the equation for f_2 (equation (9a) in his paper) should read

$$f_2''' + f_0 f_2'' + 5f_2 f_0'' - 4f_0' f_2' + 3f_1 f_1'' - 2f_1' f_1' + 2\theta_1 = 0.$$

This would account for the difference in Szewczyk's and the present solution for f_2 and θ_2 (see table A which is obtainable from the editor). (The values of f_1', θ_1, f_2' and θ_2 for the case $\sigma = 1$ are given in table A.)

The series expansions (6) and (7) enable us to calculate the values of various flow parameters near the leading edge. We obtain series for the skin friction coefficient

$$\tau_w = \left(\frac{\nu}{Ug\beta\Delta T}\right)^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_{y=0},$$

the heat transfer coefficient

$$Q = -\left(\frac{\nu U}{g\beta\Delta T}\right)^{\frac{1}{2}}\frac{1}{\Delta T}\left(\frac{\partial T}{\partial y}\right)_{y=0},$$

the momentum thickness

$$\delta_2 = \left(\frac{g\beta\Delta T}{\nu U}\right)^{\frac{1}{2}}\int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)dy$$

and the temperature thickness

$$\delta_T = \left(\frac{g\beta\Delta T}{\nu U}\right)^{\frac{1}{2}}\int_0^\infty \frac{u}{U}\left(\frac{T - T_0}{\Delta T}\right)dy.$$

For the case $\sigma = 1$, we get

$$\tau_w = (2\xi)^{-\frac{1}{2}}(0{\cdot}4696 \pm 1{\cdot}6216\xi - 1{\cdot}2699\xi^2 \pm \ldots),$$

$$Q = (2\xi)^{-\frac{1}{2}}(0{\cdot}4696 \pm 0{\cdot}3834\xi - 0{\cdot}6544\xi^2 \pm \ldots),$$

$$\delta_2 = (2\xi)^{\frac{1}{2}}(0{\cdot}4696 \pm 0{\cdot}2706\xi + 0{\cdot}0632\xi^2 \pm \ldots),$$

$$\delta_T = (2\xi)^{\frac{1}{2}}(0{\cdot}4696 \pm 0{\cdot}1278\xi - 0{\cdot}1309\xi^2 \pm \ldots).$$

4. Asymptotic expansion (favourable case only)

A long way from the leading edge, the boundary layer is formed mainly by the buoyancy forces. This suggests the following transformation,

$$\psi = 4\nu C x^{\frac{3}{4}} \phi(\xi, \bar{\eta}), \quad T - T_0 = \Delta T \bar{\theta}(\xi, \bar{\eta}),$$

where $$\bar{\eta} = Cy/x^{\frac{1}{4}},$$

and $$C = (g\beta\Delta T/4\nu^2)^{\frac{1}{4}}.$$

The boundary-layer equations (2) and (3) become

$$\frac{\partial^3\phi}{\partial\bar{\eta}^3} + \bar{\theta} + 3\phi\frac{\partial^2\phi}{\partial\bar{\eta}^2} - 2\left(\frac{\partial\phi}{\partial\bar{\eta}}\right)^2 = 4\xi\left(\frac{\partial\phi}{\partial\bar{\eta}}\frac{\partial^2\phi}{\partial\bar{\eta}\partial\xi} - \frac{\partial\phi}{\partial\xi}\frac{\partial^2\phi}{\partial\bar{\eta}^2}\right), \tag{8}$$

$$\frac{1}{\sigma}\frac{\partial^2\bar{\theta}}{\partial\bar{\eta}^2} + 3\phi\frac{\partial\bar{\theta}}{\partial\bar{\eta}} = 4\xi\left(\frac{\partial\phi}{\partial\bar{\eta}}\frac{\partial\bar{\theta}}{\partial\xi} - \frac{\partial\phi}{\partial\xi}\frac{\partial\bar{\theta}}{\partial\bar{\eta}}\right), \tag{9}$$

with boundary conditions

$$\phi = \frac{\partial\phi}{\partial\bar{\eta}} = 0, \quad \bar{\theta} = 1 \quad \text{on} \quad \bar{\eta} = 0,$$

$$\frac{\partial\phi}{\partial\bar{\eta}} \to 2\xi^{-\frac{1}{2}}, \quad \bar{\theta} \to 0 \quad \text{as} \quad \bar{\eta} \to \infty.$$

The form of the outer boundary condition on ϕ suggests expansions of ϕ and $\bar{\theta}$ in the form

$$\phi = \phi_0(\bar{\eta}) + \xi^{-\frac{1}{2}}\phi_1(\bar{\eta}) + \xi^{-1}\log\xi\,\Phi_2(\bar{\eta}) + \xi^{-1}\phi_2(\bar{\eta}) + \ldots, \tag{10}$$

$$\bar{\theta} = \bar{\theta}_0(\bar{\eta}) + \xi^{-\frac{1}{2}}\bar{\theta}_1(\bar{\eta}) + \xi^{-1}\log\xi\,H_2(\bar{\eta}) + \xi^{-1}\bar{\theta}_2(\bar{\eta}) + \ldots. \tag{11}$$

Terms of $O(\xi^{-1}\log\xi)$ have to be included in the expansions (10) and (11) in order to be able to solve the equations for the terms of $O(\xi^{-1})$. The necessity for including logarithmic terms in asymptotic expansions in boundary-layer theory is discussed by Stewartson (1957, 1964).

The equations for ϕ_0, $\bar{\theta}_0$, ϕ_1 and $\bar{\theta}_1$ are the same as those given by Szewczyk (equations (23) and (24) in his paper). The numerical solutions for ϕ_0 and $\bar{\theta}_0$ are given by Ostrach (1953) and the values of ϕ_1' and $\bar{\theta}_1$ for the case $\sigma = 1$ are given in table *B* which is available from the editor.

The equations for the terms of $O(\xi^{-1}\log\xi)$ are

$$\Phi_2''' + H_2 + 3\phi_0\Phi_2'' - \phi_0''\Phi_2 = 0, \tag{12}$$

$$\frac{1}{\sigma}H_2'' + 3\phi_0 H_2' - \Phi_2\bar{\theta}_0' + 4\phi_0' H_2 = 0, \tag{13}$$

with boundary conditions

$$\Phi_2(0) = \Phi_2'(0) = \Phi_2'(\infty) = H_2(0) = H_2(\infty) = 0.$$

The solution of (12) and (13) which satisfies the required boundary conditions is

$$\Phi_2 = \lambda(\bar{\eta}\phi_0' - 3\phi_0) = \lambda\Phi_c, \quad H_2 = \lambda\bar{\eta}\bar{\theta}_0' = \lambda H_c,$$

where λ is an, as yet, undetermined constant.

The equations for the terms of $O(\xi^{-1})$ then become

$$\phi_2''' + \bar{\theta}_2 + 3\phi_0\phi_2'' - \phi_0''\phi_2 = 4\lambda(3\phi_0\phi_0'' - 2\phi_0'^2) - \phi_1\phi_1'', \tag{14}$$

$$\frac{1}{\sigma}\bar{\theta}_2'' + 3\phi_0\bar{\theta}_2' - \phi_2\bar{\theta}_0' + 4\phi_0'\bar{\theta}_2 = -\bar{\theta}_1'\phi_1 - 2\phi_1'\bar{\theta}_1 + 12\lambda\bar{\theta}_0'\phi_0, \tag{15}$$

with boundary conditions

$$\phi_2(0) = \phi_2'(0) = \bar{\theta}_2(0) = \phi_2'(\infty) = \bar{\theta}_2(\infty) = 0.$$

Szewczyk omitted the terms of $O(\xi^{-1}\log\xi)$ and his solution for the terms of $O(\xi^{-1})$ appears to be wrong. λ has to be chosen so that we can obtain solutions for (14) and (15) which satisfy the required boundary conditions. The method of doing this is discussed in the appendix. We find that, for $\sigma = 1$,

$$\lambda = -0{\cdot}015643.$$

(The values of Φ_2' and H_2 are given in table B.) We can still add arbitrary multiples $\mu\Phi_c$ and μH_c to any solution of (14) and (15) and this will still satisfy the required boundary conditions. μ will depend on some overall property of the boundary layer which we are unable to determine. If, however, we compare the velocity and temperature profiles obtained from the asymptotic series with those obtained from the step-by-step solution, a value of $\mu = 0{\cdot}03 \pm 0{\cdot}01$ is suggested.

From expansions (10) and (11) we can determine the asymptotic values of the flow parameters. We find, for $\sigma = 1$,

$$\tau_w = 2^{\frac{1}{2}}\xi^{\frac{1}{4}}(0{\cdot}6422 + 0{\cdot}0830\xi^{-\frac{1}{2}} + 0{\cdot}0105\xi^{-1}\log\xi + (0{\cdot}0974 - 0{\cdot}6422\mu)\xi^{-1} + \ldots),$$

$$Q = 2^{-\frac{1}{2}}\xi^{-\frac{1}{4}}(0{\cdot}5671 + 0{\cdot}0712\xi^{-\frac{1}{2}} - 0{\cdot}0089\xi^{-1}\log\xi + 0{\cdot}5671\mu\xi^{-1} + \ldots),$$

$$\delta_2 = 2^{\frac{3}{2}}\xi^{\frac{3}{4}}(-0{\cdot}1839 + 0{\cdot}1832\xi^{-\frac{1}{2}} - 0{\cdot}0144\xi^{-1}\log\xi + \ldots),$$

$$\delta_T = 2^{\frac{3}{2}}\xi^{\frac{3}{4}}(0{\cdot}1890 + 0{\cdot}0772\xi^{-\frac{1}{2}} + 0{\cdot}0089\xi^{-1}\log\xi + \ldots).$$

5. Step-by-step solution

The step-by-step solution is an accurate one, i.e. the full boundary-layer equations are solved and the accuracy is limited only by the time taken to perform the calculations on the computer. We follow closely the method that Terrill (1960) used to describe the flow caused by a retarded main stream. The idea is that, knowing velocity and temperature profiles at one station ξ_1, we can calculate them at another station ξ_2, downstream of ξ_1. As in Terrill's case we first apply a transformation to the boundary-layer equations (1), (2) and (3). For starting the numerical solution, the transformed equations (4) and (5) are the appropriate ones to use. In the favourable case this transformation is not appropriate far downstream, and we have to use the transformed equations (8) and (9); the change over from one set of equations to the other being done most conveniently at $\xi = 1$. In this case we proceed with the integration in the ξ-direction until the numerical solution agrees with the asymptotic series solution to the required accuracy. In the adverse case the flow separates and so the numerical method cannot be used beyond this point.

We will give an outline of the method used to solve (4) and (5). A similar method was used on (8) and (9); and a full description of the methods used will be given in my Ph.D. thesis (Manchester University).

As in Terrill's case we work in terms of $q = \partial f/\partial \eta$. The first approximation to be made is to replace derivatives in the ξ-direction by differences and all other quantities by averages. This leads to a fifth-order system of non-linear ordinary differential equations which we have to solve by iteration. The iterative procedure used was as follows: if q_1, θ_1 and q_2, θ_2 are the values of velocity and temperature at ξ_1 and ξ_2 respectively, then we define

$$v = q_1 + q_2 \quad \text{and} \quad u = \theta_1 + \theta_2.$$

Suppose $v^{(m)}$ and $u^{(m)}$ are the mth iterative approximation then $v^{(m+1)}$ and $u^{(m+1}$ are given by the equations

$$\frac{d^2 v^{(m+1)}}{d\eta^2} + \frac{dv^{(m)}}{d\eta}\int_0^\eta \left(\frac{v^{(m+1)}}{2} + \lambda(v^{(m+1)} - 2q_1)\right) d\eta$$
$$\pm (\xi_1 + \xi_2) u^{(m)} - \lambda v^{(m)}(v^{(m+1)} - 2q_1) = 0, \tag{16}$$

$$\frac{1}{\sigma}\frac{d^2 u^{(m+1)}}{d\eta^2} + \frac{du^{(m+1)}}{d\eta}\int_0^\eta \left(\frac{v^{(m)}}{2} + \lambda(v^{(m)} - 2q_1)\right) d\eta$$
$$+ \lambda v^{(m)}(u^{(m+1)} - 2\theta_1) = 0, \tag{17}$$

where $v^{(m)}$ in (17) is the $v^{(m+1)}$ calculated from (16), and

$$\lambda = \frac{\xi_2 + \xi_1}{\xi_2 - \xi_1}.$$

This iteration process was found to converge easily.

To solve (16) and (17), differences are introduced in the η-direction. This reduces the problem to the solution of the two matrix equations of the form

$$A^{(m)}\mathbf{v}^{(m+1)} = \mathbf{c}^{(m)}, \tag{18}$$

$$B^{(m)}\mathbf{u}^{(m+1)} = \mathbf{d}^{(m)}, \tag{19}$$

where the elements of the column vectors $\mathbf{c}^{(m)}$ and $\mathbf{d}^{(m)}$ are all known; $A^{(m)}$ is the same matrix as that given by Terrill, and $B^{(m)}$ is a band matrix. The method of Choleski (Hartree 1958, pp. 180–4) is used to solve (18) and (19). Using this method and the special form of the matrices $A^{(m)}$ and $B^{(m)}$ we can keep the storage space needed on the computer down to a minimum.

The program was initially written to achieve an overall accuracy of 4 decimals in q and θ. The errors arising from using finite differences in the ξ-direction were kept small by covering the step from ξ_1 to ξ_2 in first one, then two steps, and ensuring that the maximum modulus of the difference in the two solutions thus obtained was less than 5×10^{-5}. The values of q and θ obtained from integrating at the half intervals were the ones printed out and also the ones used for the next step. The integrations in the η-direction were carried out with η taking the values

$$\eta = 0{\cdot}05(0{\cdot}05)\,6{\cdot}4$$

in the favourable case, and $\eta = 0{\cdot}05(0{\cdot}05)\,7{\cdot}2$

in the adverse case. For the solution of (8) and (9), $\bar{\eta}$ took the values

$$\bar{\eta} = 0{\cdot}05(0{\cdot}05)\,10.$$

It was found that the outer boundary conditions on q and θ were satisfied to the required accuracy at the points chosen.

The size of the truncation errors in the η-direction were checked using finite difference estimates for the errors, and these showed that the required accuracy had been reached. In the favourable case the truncation errors were also checked by performing another integration with η taking the values

$$\eta = 0{\cdot}025(0{\cdot}025)\,3{\cdot}2.$$

The outer boundary conditions on q and θ were calculated from the solution with step length 0·05. This also confirmed that the required accuracy had been reached.

The initial profiles were taken as $f_0'(\eta)$ and $\theta_0'(\eta)$ from the series solution for small ξ. The numerical solution could not start at $\xi = 0$ because the iteration process did not converge there, but had to start from $\xi = 5\times 10^{-6}$, with an initial small step of 5×10^{-6}, so a procedure was introduced for doubling the step length when the maximum number of iterations needed in going from ξ_1 to ξ_2 was less than four.

From the velocity and temperature profiles calculated at a particular station we can include a procedure for calculating the flow parameters. δ_2 and δ_T are calculated using the Euler–Maclarin formula to calculate the integrals. From the boundary-layer equations, we know that, on $\eta = 0$,

$$\partial^2 q/\partial\eta^2 = \mp 2\xi, \quad \partial^3 q/\partial\eta^3 = \mp 2\xi\partial\theta/\partial\eta,$$
$$\partial^2\theta/\partial\eta^2 = 0, \quad \partial^3\theta/\partial\eta^3 = 0.$$

Putting these results in the Taylor series expansions for q and θ, for fixed ξ, we get the formulae

$$\left(\frac{\partial\theta}{\partial\eta}\right)_0 = \frac{\theta_1 - 1}{h} + O(h^3), \tag{20a}$$

$$\left(\frac{\partial\theta}{\partial\eta}\right)_0 = \frac{16\theta_1 - \theta_2 - 15}{14h} + O(h^4), \tag{20b}$$

$$\left(\frac{\partial q}{\partial\eta}\right)_0 = \frac{q_1}{h} \pm \xi h \pm \frac{\xi h^2}{3}\left(\frac{\partial\theta}{\partial\eta}\right)_0 + (Oh^3), \tag{21a}$$

$$\left(\frac{\partial q}{\partial\eta}\right)_0 = \frac{16q_1 - q_2 \pm 12\xi h^2 \pm \frac{8}{3}\xi h^3\left(\frac{\partial\theta}{\partial\eta}\right)_0}{14h} + O(h^4). \tag{21b}$$

The results from using the formulae (20a) and (20b) and from (21a) and (21b) were compared and a difference of at most 2 in the fifth decimal place resulted, except in the adverse case when the separation point was approached. In this case the difference was still small, being at most 3 in the fourth decimal place. Similar formulae were used in the calculation for $\xi \geqslant 1$ (favourable case).

6. Discussion of the results

Tables 1 and 2 give the flow parameters at various values of ξ for the favourable and adverse cases respectively. Tables of velocity and temperature profiles for both cases will be included in my Ph.D. thesis (Manchester University).

The initial integration in the adverse case, working to an accuracy of 4 figures, showed that, at separation, τ_w and Q become singular. The integration in this case was then performed again, working to an accuracy of 5 figures. Graphs of τ_w and Q are given in figure 1. From them it appears that as $\xi \to \xi_s$ (ξ_s is the point where the flow separates)

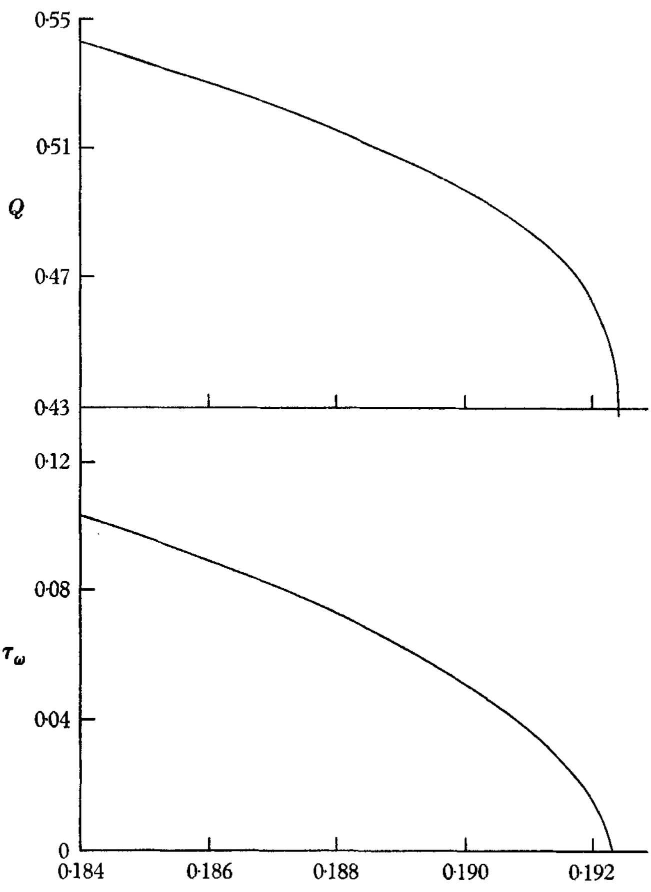

FIGURE 1. The behaviour of the skin friction τ_ω and the heat transfer Q near separation.

$$\tau_w \to 0, \quad Q \to Q_s,$$

but
$$d\tau_w/d\xi \to \infty \quad \text{and} \quad dQ/d\xi \to \infty,$$

where
$$\xi_s = 0{\cdot}192357$$

and
$$Q_s = 0{\cdot}428.$$

ξ	Q	τ_ω	δ_2	δ_T
0·00001	1·050 α2	1·050 α2	2·100 α − 3	2·100 α − 3
0·00448	4·978 α0	5·038 α0	4·433 α − 2	4·449 α − 2
0·04928	1·552 α0	1·742 α0	1·433 α − 1	1·493 α − 1
0·10048	1·122 α0	1·388 α0	1·985 α − 1	2·157 α − 1
0·20288	8·135 α − 1	1·196 α0	2·655 α − 1	3·131 α − 1
0·30528	7·066 α − 1	1·142 α0	3·059 α − 1	3·912 α − 1
0·46912	6·011 α − 1	1·120 α0	3·412 α − 1	4·974 α − 1
0·55104	5·672 α − 1	1·121 α0	3·498 α − 1	5·452 α − 1
0·67392	5·284 α − 1	1·127 α0	3·543 α − 1	6·124 α − 1
0·79680	4·989 α − 1	1·137 α0	3·506 α − 1	6·754 α − 1
0·87872	4·827 α − 1	1·145 α0	3·443 α − 1	7·156 α − 1
1·00000	4·624 α − 1	1·157 α0	3·301 α − 1	7·729 α − 1
1·2815	4·266 α − 1	1·186 α0	2·787 α − 1	8·977 α − 1
1·5375	4·028 α − 1	1·213 α0	2·135 α − 1	1·004 α0
2·1007	3·659 α − 1	1·268 α0	2·269 α − 2	1·220 α0
2·5103	3·468 α − 1	1·304 α0	− 1·485 α − 1	1·365 α0
4·0463	3·018 α − 1	1·419 α0	− 9·649 α − 1	1·860 α0
5·2751	2·799 α − 1	1·494 α0	− 1·765 α0	2·217 α0
10·190	2·334 α − 1	1·714 α0	− 5·794 α0	3·462 α0
19·201	1·970 α − 1	1·974 α0	− 1·529 α1	5·380 α0
29·032	1·767 α − 1	2·171 α0	− 2·761 α1	7·209 α0
51·969	1·512 α − 1	2·491 α0	− 6·144 α1	1·094 α1
78·184	1·37 α − 1	2·747 α0	− 1·061 α2	1·471 α1
150·27	1·157 α − 1	3·218 α0	− 1·161 α2	1·471 α1
242·02	1·025 α − 1	3·615 α0	− 4·632 α2	3·365 α1
346·88	9·352 α − 2	3·949 α0	− 7·353 α2	4·390 α1
504·17	8·507 α − 2	4·331 α0	− 1·186 α3	5·790 α1
102·8	7·107 α − 2	5·615 α0	− 2·933 α3	9·830 α1
1343·0	6·645 α − 2	5·519 α0	− 4·111 α3	1·199 α1
series	6·647 α − 2	5·518 α0	− 4·116 α3	1·198 α1

TABLE 1. Flow parameters (favourable case)—given in floating point notation.

ξ	Q	τ_ω	δ_2	δ_T
0·00249	7·324	7·284	0·0301	0·0304
0·01265	3·099	3·006	0·0793	0·0725
0·03174	1·813	1·654	0·1205	0·1172
0·08499	1·045	0·7764	0·2033	0·1886
0·11366	0·8664	0·5492	0·2390	0·2158
0·14234	0·7321	0·3676	0·2719	0·2387
0·16691	0·6288	0·2224	0·2986	0·2554
0·17920	0·5271	0·1433	0·3117	0·2678
0·18739	0·5222	0·0782	0·3207	0·2672
0·19098	0·4852	0·0366	0·3240	0·2691
0·19174	0·4703	0·0289	0·3248	0·2694
0·19200	0·4628	0·0168	0·3251	0·2696
0·19226	0·4499	0·0081	0·3253	0·2697
0·192337	0·4399	0·0032	0·3254	0·2697
0·192353	0·4340	0·0012	0·3254	0·2697
0·192355	0·4320	0·0008	0·3254	0·2697
0·192356	0·4293	0·0002	0·3254	0·2697
0·192357	0·4281	0·0000	0·3254	0·2697

TABLE 2. Flow parameters (adverse case).

The existence of this singularity conflicts with a result of Stewartson (1962), who gave an argument to show that the solution should be regular at separation. Actually, Stewartson proved his result for a compressible boundary layer with heat transfer, but the same theory applies in this case.

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Appendix. Determination of λ

The solutions ϕ_2 and $\overline{\theta}_2$ of (14) and (15) are of the form

$$\phi_2 = \phi_{2a} + 4\lambda\phi_{2b} + \mu\Phi_c, \quad \overline{\theta}_2 = \overline{\theta}_{2a} + 4\lambda\overline{\theta}_{2b} + \mu H_c,$$

where μ is an undetermined constant, ϕ_{2a} and $\overline{\theta}_{2a}$ satisfy (14) and (15) with $\lambda = 0$, and ϕ_{2b} and $\overline{\theta}_{2b}$ satisfy

$$\phi'''_{2b} + \overline{\theta}_{2b} + 3\phi_0\phi''_{2b} - \phi''_0\phi_{2b} = 3\phi_0\phi''_0 - 2\phi'^2_0,$$

$$\frac{1}{\sigma}\overline{\theta}''_{2b} + 3\phi_0\overline{\theta}'_{2b} - \overline{\theta}'_0\phi_{2b} + 4\phi'_0\overline{\theta}_{2b} = 3\phi_0\overline{\theta}'_0.$$

We now construct solutions of the form

$$\phi_{2a} = X_a + \alpha X_1 + \beta X_2, \quad \overline{\theta}_{2a} = W_a + \alpha W_1 + \beta W_2,$$

$$\phi_{2b} = X_b + \gamma X_1 + \delta X_2, \quad \overline{\theta}_{2b} = W_b + \gamma W_1 + \delta W_2,$$

where (X_a, W_a) and (X_b, W_b) are particular integrals of the respective equations with

$$X''_a(0) = 0, \quad X''_b(0) = 0,$$

$$W'_a(0) = 0, \quad W'_b(0) = 0,$$

and (X_1, W_1), (X_2, W_2) are complementary functions with

$$X''_1(0) = 1, \quad W'_1(0) = 0, \quad X''_2(0) = 0, \quad W'_2(0) = 1.$$

From the equations, we have, as $\overline{\eta} \to \infty$,

$$W_i \to Q_i,$$

$$X'_i \sim -\frac{Q_i\overline{\eta}}{3\phi_0(\infty)} + P_i,$$

where $i = a, b, 1, 2$ and Q_i, P_i are constants. Now Φ_c and H_c are complementary functions with

$$\Phi''_c(0) = -\phi''_0(0), \quad H'_c(0) = \overline{\theta}'_0(0),$$

so that $\quad \Phi_c = -\phi''_0(0)X_1 + \overline{\theta}'_0(0)X_2, \quad H_c = -\phi''_0(0)W_1 + \overline{\theta}'_0(0)W_2$

and as Φ'_c and $H_c \to 0$ as $\overline{\eta} \to \infty$, we get the relations

$$\overline{\theta}'_0(0)Q_2 = \phi''_0(0)Q_1, \quad \overline{\theta}'_0(0)P_2 = \phi''_0(0)P_1. \tag{22}$$

We also have

$$\overline{\theta}_{2a}(\infty) = Q_a + \alpha Q_1 + \beta Q_2,$$

$$\phi'_{2a}(\overline{\eta}) \sim P_a + \alpha P_1 + \beta P_2 - (\overline{\theta}_{2a}(\infty)\overline{\eta}/3\phi_0(\infty)).$$

From this we can see why the term of $O(\xi^{-1}\log\xi)$ had to be included. Trying to make $\phi'_{2a}(\infty)$ and $\bar{\theta}_{2a}(\infty) = 0$ gives equations for α and β which become, on using (22),

$$\alpha + (\phi''_0(0)/\bar{\theta}'_0(0))\beta = -P_a/P_1, \quad \alpha + (\phi''_0(0)/\bar{\theta}'_0(0))\beta = -Q_a/Q_1.$$

These are inconsistent, so we have to consider the forms of $\bar{\theta}_2$ and ϕ'_2 as $\bar{\eta} \to \infty$

$$\bar{\theta}_2(\infty) = Q_a + 4\lambda Q_b + (\alpha + 4\lambda\gamma)\,Q_1 + (\beta + 4\lambda\,\delta)\,Q_2,$$

$$\phi'_2(\bar{\eta}) \sim P_a + 4\lambda P_b + (\alpha + 4\lambda\gamma)\,P_1 + (\beta + 4\lambda\delta)\,P_2 - (\bar{\theta}_2(\infty)\bar{\eta}/3\phi_0(\infty)).$$

Putting $\bar{\theta}_2(\infty) = \phi'_2(\infty) = 0$ leads to the equations, using (22),

$$(\alpha + 4\lambda\gamma) + \frac{\phi''_0(0)}{\bar{\theta}'_0(0)}(\beta + 4\lambda\delta) = -\left(\frac{P_a + 4\lambda P_b}{P_1}\right), \tag{23}$$

$$(\alpha + 4\lambda\gamma) + \frac{\phi''_0(0)}{\bar{\theta}'_0(0)}(\beta + 4\lambda\delta) = -\left(\frac{Q_a + 4\lambda Q_b}{Q_1}\right), \tag{24}$$

which are consistent provided

$$4\lambda = \frac{P_1 Q_a - P_a Q_1}{P_b Q_1 - Q_b P_1}.$$

A numerical integration of the equations shows that, for $\sigma = 1$,

$$\lambda = -0{\cdot}015643.$$

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